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D-Dimensional Ideal Gas in Parastatistics: Thermodynamic Properties

M. C. de Sousa Vieira[1] **and C. Tsallis**[1]

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We consider a parastatistics ideal gas with energy spectrum $\varepsilon \propto |\mathbf{k}|^{\alpha}$ ($\alpha > 0$) or even more generally in a d-dimensional box with volume V (periodic boundary conditions), the number N of the gas particles being well determined (real particles) or not (quasiparticles). We calculate the main thermodynamic quantities (chemical potential, internal energy, specific heat C, equation of state, latent heat, average numbers of particles) for arbitrary d, α, T (temperature), and p (maximal number of particles per state allowed in the parastatistics). The main asymptotic regimes are worked out explicitly. In particular, the Bose–Einstein condensation for fixed density N/V appears as a nonuniform convergence in the $p \to \infty$ limit, in complete analogy with the standard critical phenomena that appear in interacting systems in the $N \to \infty$ limit. The system behaves essentially like a Fermi–Dirac one for *all* finite values of p, and reveals a Bose–Einstein behavior *only* in the $p \to \infty$ limit. For instance, at low temperatures $C \propto T$ if $p < \infty$ and $C \propto T^{d/\alpha}$ if $p \to \infty$. Finally, the Sommerfeld integral and its expansion are generalized to an arbitrary, finite p.

KEY WORDS: Parastatistics; ideal gas; Bose–Einstein condensation; Sommerfeld integral.

1. INTRODUCTION

Since the pioneering work by Gentile,[1] many different studies have been done which interpolate between Bose–Einstein and Fermi–Dirac statistics (see, for instance, Ref. 2). Possible applications have been sought in field theory and elementary particle physics[3–6] as well as in condensed matter physics (molecular excitons and magnons,[7] quantum Hall effect[8]). The ideal gas in parastatistics naturally constitutes a privileged reference system, which can be used as unperturbed starting point to study more

[1] Centro Brasileiro de Pesquisas Físicas/CNPq, Rua Dr. Xavier Sigaud, 150, 22290-Rio de Janeiro-RJ, Brazil.

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complex systems. Also it will be shown that many exact (or asymptotically exact) analytical expressions can be obtained, which could serve as testing material for various approximation methods. The purpose of the present paper is the establishment of the main thermodynamic properties of such a gas at fixed volume in d dimensions ($d>0$), and for a quite general energy spectrum $\varepsilon=(\sum_{j=1}^{d} a_j k_j^2)^{\alpha/2}$ ($a_j>0$ and $\alpha>0$; $\alpha=1$ corresponds to photons and to short-wave-vector acoustic phonons, among others; $\alpha=2$ corresponds to nonrelativistic free particles and to short wave-vector acoustic magnons in Heisenberg ferromagnets, among others).

In Section 2 we introduce the specific ideal gas model and obtain the associated density of states; in Sections 3 and 4 we calculate the main thermodynamic quantities respectively corresponding to fixed (real particles) and unfixed (quasiparticles) total number of gas particles; finally, we conclude in Section 5. In the Appendix we generalize to parastatistics the Sommerfeld integral and its standard expansion.

2. PARASTATISTICS IDEAL GAS. DENSITY OF STATES

We consider an ideal gas of N particles (or quasiparticles) in a d-dimensional box. Each particle behaves as a planar wave with energy spectrum given by

$$\varepsilon=\left(\sum_{j=1}^{d} a_j k_j^2\right)^{\alpha/2} \tag{1}$$

where $a_j>0$ $\forall j$, $\alpha>0$, and k_j is the jth component of the wave vector $\mathbf{k}$. The particular case $a_j=a$ $\forall j$ yields $\varepsilon\propto k^\alpha$ ($k\equiv|\mathbf{k}|$). Periodic boundary conditions are considered on the box, which is assumed to be an orthogonal hyperparallelepiped with side lengths $\{L_j\}$ and volume $V\equiv\prod_{j=1}^{d} L_j$. The possible wave vectors are given by

$$k_j=(2\pi/L_j)n_j, \qquad n_j=0, \pm 1, \pm 2,..., \forall j \tag{2}$$

which inserted into Eq. (1) yields

$$\sum_{j=1}^{d}\frac{(2\pi)^2 a_j}{\varepsilon^{2/\alpha}L_j^2}n_j^2=1 \tag{3}$$

This is the equation of a hyperellipsoid whose volume provides the number of states $\phi(\varepsilon)$ with energy is equal to or lower than ε. Consequently,

$$\phi(\varepsilon)=(\alpha/d)(\varepsilon/\varepsilon_0)^{d/\alpha} \tag{4}$$

with

$$\varepsilon_0 \equiv \left[\frac{2^d \pi^{d/2} \Gamma(d/2+1) (\prod_{j=1}^{d} a_j)^{1/2}}{Vd/\alpha} \right]^{\alpha/d} \tag{4'}$$

The density of states $\rho(\varepsilon)$ is therefore given by

$$\rho(\varepsilon) = \frac{d\phi(\varepsilon)}{d\varepsilon} = \frac{1}{\varepsilon_0} \left(\frac{\varepsilon}{\varepsilon_0} \right)^{d/\alpha - 1} \tag{5}$$

Notice that we treat $\phi(\varepsilon)$ as if it were a "soft" function of ε: this is correct in the thermodynamic limit in which we are interested ($N \to \infty$, $V \to \infty$, $N/V \to \text{const}$, $\varepsilon_0 \to 0$). The parastatistics thermal equilibrium average number $f(\varepsilon)$ of particles per state is given by[1]

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} - \frac{p+1}{e^{(p+1)\beta(\varepsilon - \mu)} - 1} \tag{6}$$

or alternatively by

$$f(\varepsilon) = \sum_{j=1}^{p} j e^{(p-j)\beta(\varepsilon - \mu)} \Bigg/ \sum_{j=0}^{p} e^{j\beta(\varepsilon - \mu)} \tag{6'}$$

where $\beta \equiv 1/k_B T$ is the inverse temperature, μ is the chemical potential, and p is the maximal number of particles allowed per state ($p = 1$ reproduces the Fermi–Dirac statistics and $p \to \infty$ the Bose–Einstein statistics; see Fig. 1.

3. FIXED TOTAL NUMBER OF PARTICLES (REAL PARTICLES)

3.1. Chemical Potential and Ground State Population

We consider here the total number N of gas particles as well determined. If we call respectively N_0 and N_e the populations of the ground state and the excited states, then

$$N = N_0(T) + N_e(T) \tag{7}$$

We now calculate the chemical potential $\mu(T, p)$, which is complete determined by

$$N = \int_0^\infty d\varepsilon\, \rho(\varepsilon) f(\varepsilon) \tag{8}$$

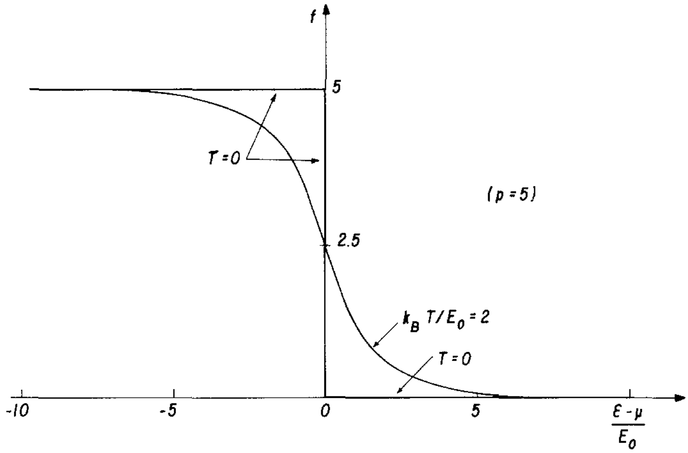


Fig. 1. Parastatistics with maximal occupation p: average population per state as a function of the energy ε measured from the chemical potential μ, for finite and vanishing temperatures. The point $(\varepsilon-\mu, f)=(0, p/2)$ is a center of symmetry.

Let us first consider the $T\to 0$ limit, in which case $f(\varepsilon)$ is the step function indicated in Fig. 1. We then have

$$N=\int_0^{\mu_0} d\varepsilon\, \rho(\varepsilon)\, p=\int_0^{\mu_0} \frac{p\varepsilon^{d/\alpha-1}}{\varepsilon_0^{d/\alpha}}\, d\varepsilon$$

$$=\frac{p}{\varepsilon_0^{d/\alpha}}\frac{\alpha}{d}\mu_0^{d/\alpha} \tag{9}$$

where $\mu_0(p)\equiv\mu(0, p)$ and we have used Eq. (5) [with $\rho(\varepsilon)\equiv 0$ for $\varepsilon<0$]. Consequently we have that

$$\mu_0(p)=\mu_0(1)/p^{\alpha/d} \tag{10}$$

where

$$\mu_0(1)=(Nd/\alpha)^{\alpha/d}\varepsilon_0 \tag{11}$$

or even

$$\mu_0(1)=\left[2^d\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right)\left(\prod_{j=1}^{d} a_j\right)^{1/2}\frac{N}{V}\right]^{\alpha/d} \tag{12}$$

where we used Eq. (4′) to make explicit the dependence of μ_0 on the concentration N/V. It is convenient to introduce the following reduced quantities:

$$t\equiv k_B T/\mu_0(1) \tag{13}$$

$$\bar{\mu}\equiv\mu/\mu_0(1) \tag{14}$$

$$x\equiv\varepsilon/\mu_0(1) \tag{15}$$

Equation (8) can therefore be rewritten as follows:

$$\frac{\alpha}{d} = \int_0^\infty dx\, x^{d/\alpha - 1} \left[\frac{1}{\exp[(x-\bar{\mu})/t] - 1} - \frac{p+1}{\exp[(p+1)(x-\bar{\mu})/t] - 1} \right] \tag{16}$$

which makes it obvious that, in energy units of $\mu_0(1)$, the chemical potential (and in fact *all* the thermodynamic quantities in which we shall be interested) depends on d and α *only* through the ratio d/α (see also Ref. 9). Through a further transformation $y \equiv x/t$ we obtain

$$\frac{\alpha}{d} t^{-d/\alpha} = \int_0^\infty dy\, y^{d/\alpha - 1} \left[\frac{1}{\exp(y - \bar{\mu}/t) - 1} - \frac{p+1}{\exp[(p+1)(y - \bar{\mu}/t)] - 1} \right] \tag{17}$$

which provides t as an explicit function of $\bar{\mu}/t$, p, and d/α. The results are indicated in Fig. 2. As can be observed there, the thermal dependence of $\bar{\mu}$,

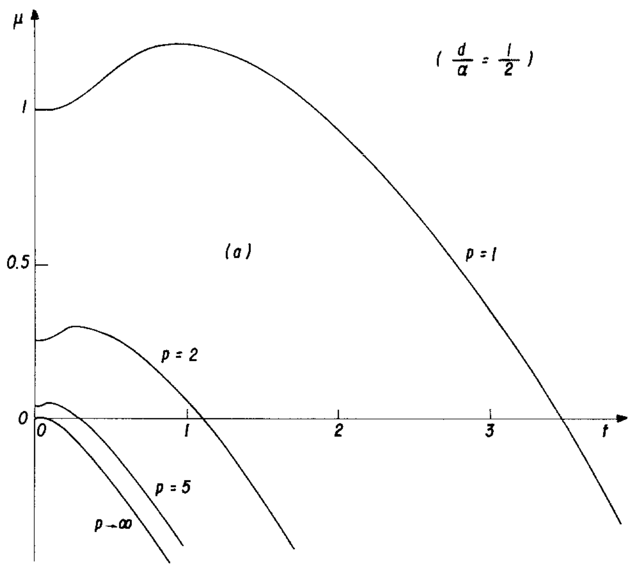


Fig. 2. Thermal dependence of the (reduced) chemical potential for typical values of p and d/α. Notice that for all $p < \infty$ the behavior is qualitatively the same (in the sense that $\bar{\mu}$ is analytic for all finite values of t) regardless of the value of d/α, and can be characterized by Fermi–Dirac ($p = 1$) behavior, whereas the $p \to \infty$ limit (Bose–Einstein) presents two different regimes, one occurring for $d/\alpha \leqslant 1$ (no Bose–Einstein condensation), the other for $d/\alpha > 1$ (Bose–Einstein condensation, due to nonanalyticity of $\bar{\mu}$ appearing at a *finite* temperature).

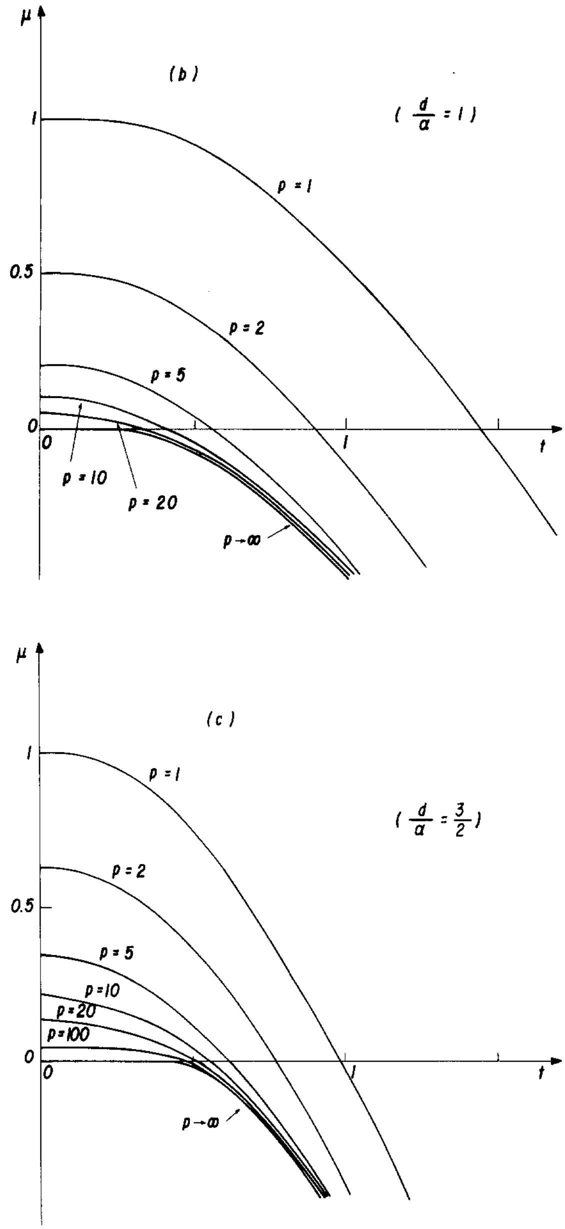


Fig. 2 (*continued*)

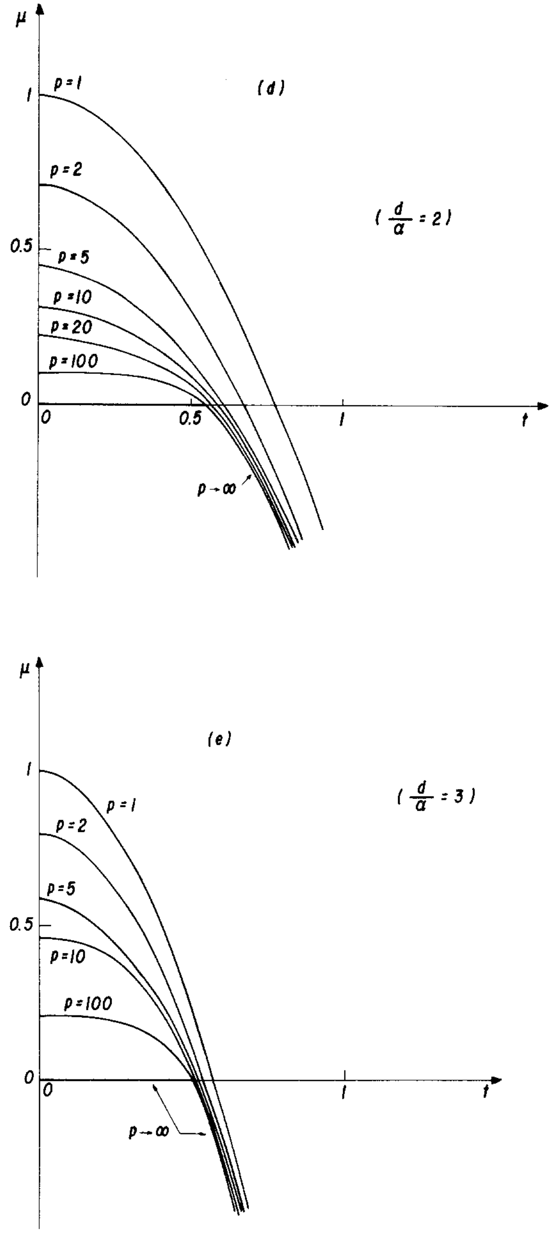


Fig. 2 (*continued*)

for fixed d/α, flattens in the low-temperature region for increasing p, and eventually exhibits, in the $p \to \infty$ limit and $d/\alpha > 1$, a "plateau" at $\bar{\mu} = 0$ (the width of the plateau increases with increasing and not too high $d/\alpha - 1$). This is of course the Bose–Einstein condensation (see Ref. 10 and references therein), and we note in the present framework that it appears in the $p \to \infty$ limit through a nonuniform convergence completly similar to those observed, in the $N \to \infty$ limit, for interacting systems presenting phase transitions.

No Bose–Einstein condensation at finite temperatures exists for $d/\alpha \leqslant 1$. In particular for $d/\alpha = 1$, this is easy to verify, since the integral of Eq. (17) is straightforwardly solved, and we obtain the following dependence of t on $\bar{\mu}/t$:

$$t = \left\{ \ln \frac{1 - \exp[(p+1)\bar{\mu}/t]}{1 - \exp(\bar{\mu}/t)} \right\}^{-1} = \left[\ln \sum_{j=0}^{p} \exp(j\bar{\mu}/t) \right]^{-1} \tag{18}$$

This expression leads, in the $p \to \infty$ limit (and taking into account that $\bar{\mu} < 0$ for $t > 0$), to the standard $d = \alpha = 2$ result, namely

$$\bar{\mu} = t \ln(1 - e^{-1/t}) \tag{19}$$

where we verify that $\bar{\mu}$ vanishes at no other place than at $t = 0$.

Let us now go back to the general expression indicated in Eq. (17), and denote by t^* the (finite) temperature at which $\bar{\mu}$ vanishes for fixed p and d/α . It follows that

$$\frac{\alpha}{d} (t^*)^{-d/\alpha} = \int_0^\infty dy\ y^{d/\alpha - 1} \left[\frac{1}{e^y - 1} - \frac{p+1}{e^{(p+1)y} - 1} \right] \tag{20}$$

The integral can be expressed (see p. 325 of Ref. 11) in terms of the Riemann zeta and gamma functions ζ and Γ, and we obtain

$$t^* = \left\{ \left[1 - \frac{1}{(p+1)^{d/\alpha - 1}} \right] \zeta\left(\frac{d}{\alpha}\right) \Gamma\left(\frac{d}{\alpha} + 1\right) \right\}^{-\alpha/d}, \quad \frac{d}{\alpha} > 1 \tag{21}$$

These results are illustrated in Fig. 3; note that $t_c(d/\alpha) \equiv \lim_{p \to \infty} t^*(p, d/\alpha)$, precisely is the Bose–Einstein condensation critical (reduced) temperature (the phase transition only exists for $p \to \infty$), and it is given by

$$t_c = \left[\frac{d}{\alpha} \int_0^\infty dy \frac{y^{d/\alpha - 1}}{e^y - 1} \right]^{-\alpha/d} \tag{22}$$

Hence

$$t_c = [\zeta(d/\alpha)\, \Gamma(d/\alpha + 1)]^{-\alpha/d} \tag{22'}$$

$$\sim \begin{cases} d/\alpha - 1 & \text{if } \ d/\alpha - 1 \to +0 \\ e\alpha/d & \text{if } \ d/\alpha \to \infty \end{cases} \tag{22''}$$

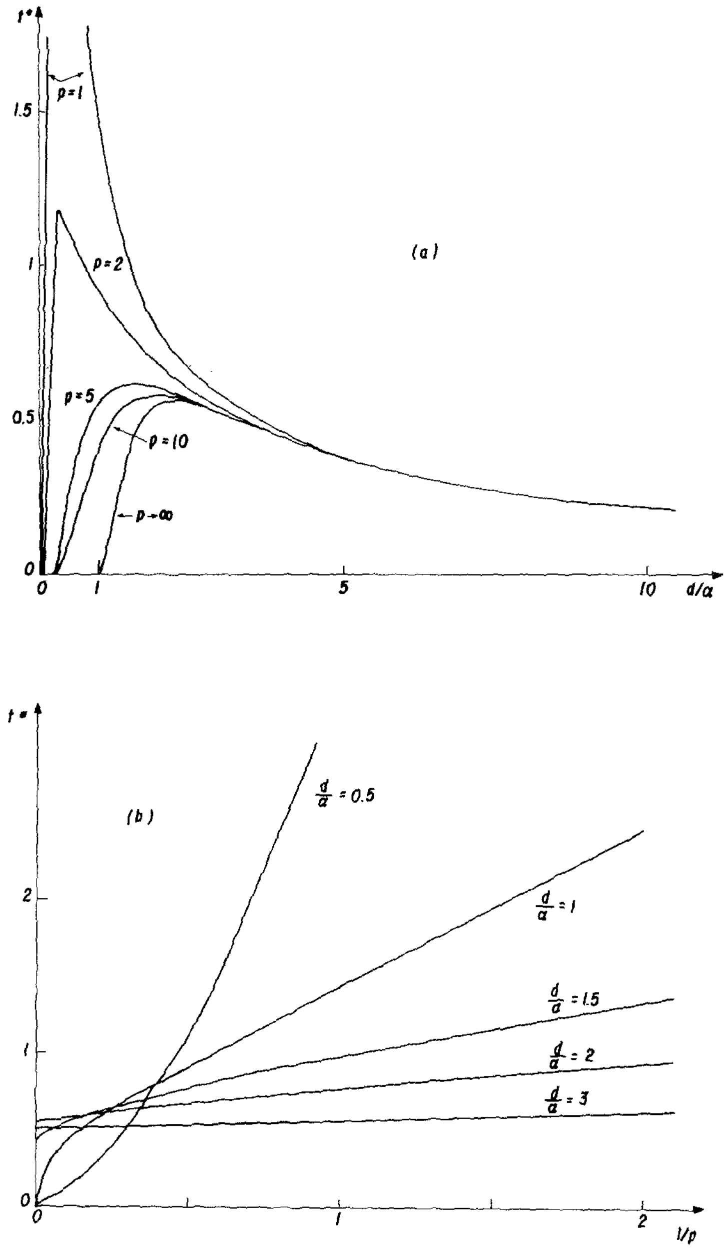


Fig. 3. Reduced temperature at which μ vanishes as a function of d/α and p. The $p \to \infty$ limit yields the d/α dependence of the (reduced) critical temperature corresponding to the Bose–Einstein condensation.

A few more steps will lead us to the thermal dependence of the (reduced) population N_0/N of the ground state below the critical temperature (above the critical temperature, N_0/N vanishes in the thermodynamic limit $N \to \infty$). We have that

$$\frac{N_0}{N} = 1 - \frac{N_e}{N} = 1 - \frac{1}{N} \int_0^\infty d\varepsilon\, \rho(\varepsilon)\, f(\varepsilon; \mu = 0)$$
$$= 1 - \frac{d}{\alpha}\, t^{d/\alpha} \int_0^\infty dy\, \frac{y^{d/\alpha - 1}}{e^y - 1}$$

Hence

$$N_0/N = 1 - (t/t_c)^{d/\alpha}, \qquad t \leqslant t_c \tag{23}$$

where we have used Eq. (22).

We now dedicate the rest of this subsection to determining the low- and high-temperature expansions of the chemical potential.

For the low-temperature asymptotic behavior we can use, in Eq. (17), the expansion (A.15) of the Appendix (with $A = 1$, $\gamma = d/\alpha - 1$, and $z = \bar{\mu}/t$), obtaining

$$\bar{\mu} = \left(p^{\alpha/d} \left\{ 1 + \sum_{m=1}^{\infty} \left\{ \frac{2d}{p\alpha} \zeta(2m) \left[1 - \frac{1}{(p+1)^{2m-1}} \right] \right. \right. \right.$$
$$\left. \left. \left. \times \prod_{j=0}^{2(m-1)} \left(\frac{d}{\alpha} - 1 - j \right) \right\} \left(\frac{t}{\bar{\mu}} \right)^{2m} \right\}^{\alpha/d} \right)^{-1} \tag{24}$$

The solution of this equation clearly has the form

$$\bar{\mu} = \frac{1}{p^{\alpha/d}} \left(1 + \sum_{r=1}^{\infty} \alpha_r t^{2r} \right) \tag{25}$$

The substitution of this expression into both sides of Eq. (24) provides, by simple identification, the knowledge of $\{\alpha_r\}$ up to any desired order. To the lowest correction we obtain

$$\bar{\mu} \sim \frac{1}{p^{\alpha/d}} \left[1 - \frac{\pi^3}{3} \frac{p^{2\alpha/d}}{p+1} \left(\frac{d}{\alpha} - 1 \right) t^2 \right], \qquad t \to 0, \qquad \forall d/\alpha \tag{26}$$

Note the change of curvature that occurs at $d/\alpha = 1$. As a matter of fact, for $d/\alpha = 1$, the departure of $\bar{\mu}$ from $1/p$ is slower than *any* power of t (and not only t^2). For this case we obtain from Eq. (18),

$$\bar{\mu} \sim \begin{cases} (1/p)(1 - t e^{-1/pt}), & t \to 0, \quad d/\alpha = 1, \quad p < \infty \qquad (27) \\ -t e^{-1/t}, & t \to 0, \quad d/\alpha = 1, \quad p \to \infty \qquad (27') \end{cases}$$

For the high-temperature asymptotic behavior we obtain from Eq. (17)

$$\bar{\mu} \sim -\frac{d}{\alpha} t \ln t - t \ln \Gamma\left(\frac{d}{\alpha}+1\right) - \frac{t^{-d/\alpha+1}}{2^{d/\alpha}\Gamma(d/\alpha+1)} + \frac{t^{-pd/\alpha+1}}{(p+1)^{d/\alpha-1}[\Gamma(d/\alpha+1)]^{p}}, \qquad t \to \infty, \quad \forall d/\alpha \tag{28}$$

Note that, as expected, the high-temperature leading terms do not depend on p (Maxwell–Boltzmann limit).

3.2. Internal Energy, Specific Heat, Equation of State, and Latent Heat

The internal energy is given by

$$U = \int_0^{\infty} d\varepsilon\, \varepsilon \rho(\varepsilon)\, f(\varepsilon) \tag{29}$$

Hence

$$U/N = \mu_0(1)\, u \tag{30}$$

with

$$u(t) \equiv \frac{d}{\alpha} t^{d/\alpha+1} \int_0^{\infty} dy\, y^{d/\alpha} \left[\frac{1}{\exp(y-\bar{\mu}/t)-1} - \frac{p+1}{\exp[(p+1)(y-\bar{\mu}/t)]-1}\right] \tag{31}$$

where we have used Eqs. (5), (9), and (11). This expression, together with $\bar{\mu}(t)$ determined in Section 3.1, completely determines the thermal dependence of u and consequently U/N. The specific heat C is given by

$$C = dU/dT \tag{32}$$

Therefore the specific heat c per particle is given by

$$c \equiv C/Nk_{\mathrm{B}} = d(U/N)/k_{\mathrm{B}}\, dT = du/dt \tag{33}$$

where we have used Eq. (30). The results are presented in Fig. 4.

Let us now discuss the low-temperature behavior of c. For finite p we can expand the integral of Eq. (31) as indicated in the Appendix, and then use Eqs. (26) and (27); we obtain

$$c \sim \frac{2\pi^2}{3} \frac{d}{\alpha} \frac{p^{d/\alpha}}{p+1}\, t, \qquad t \to 0 \tag{34}$$

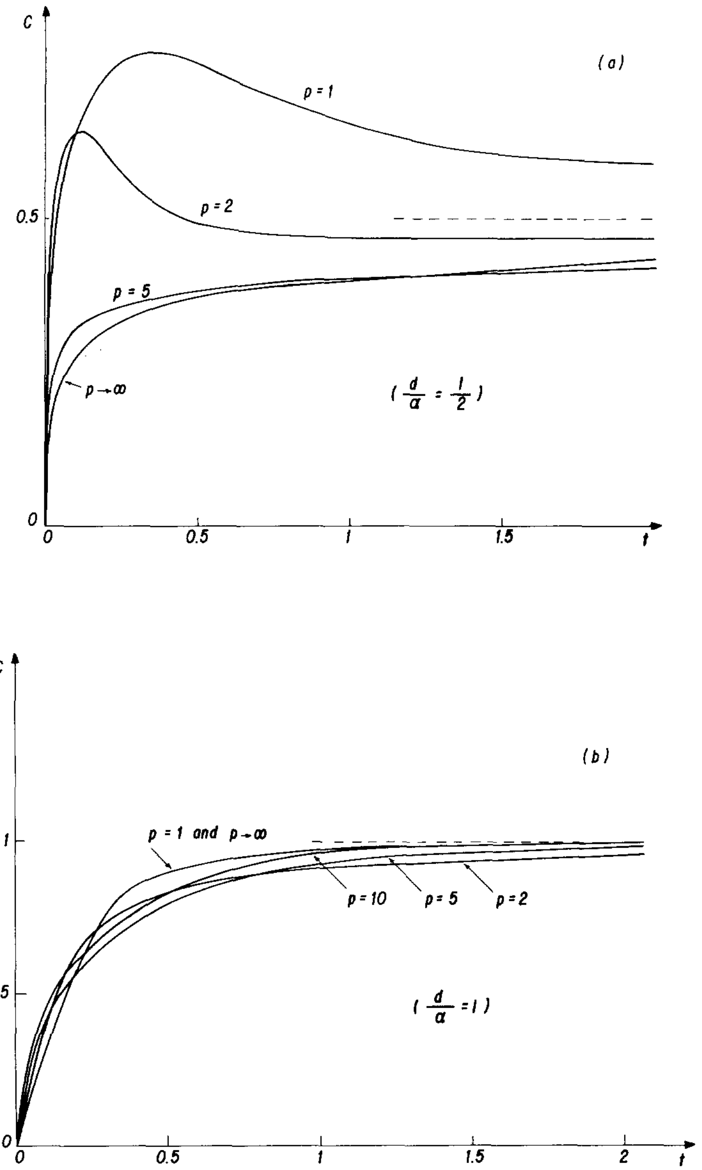


Fig. 4. (a–e) Thermal dependence of the (reduced) specific heat for typical values of p and d/α. (f) For the $p \to \infty$ limit, the d/α dependence of the height of the *cusp* that appears in versus t for $d/\alpha > 1$.

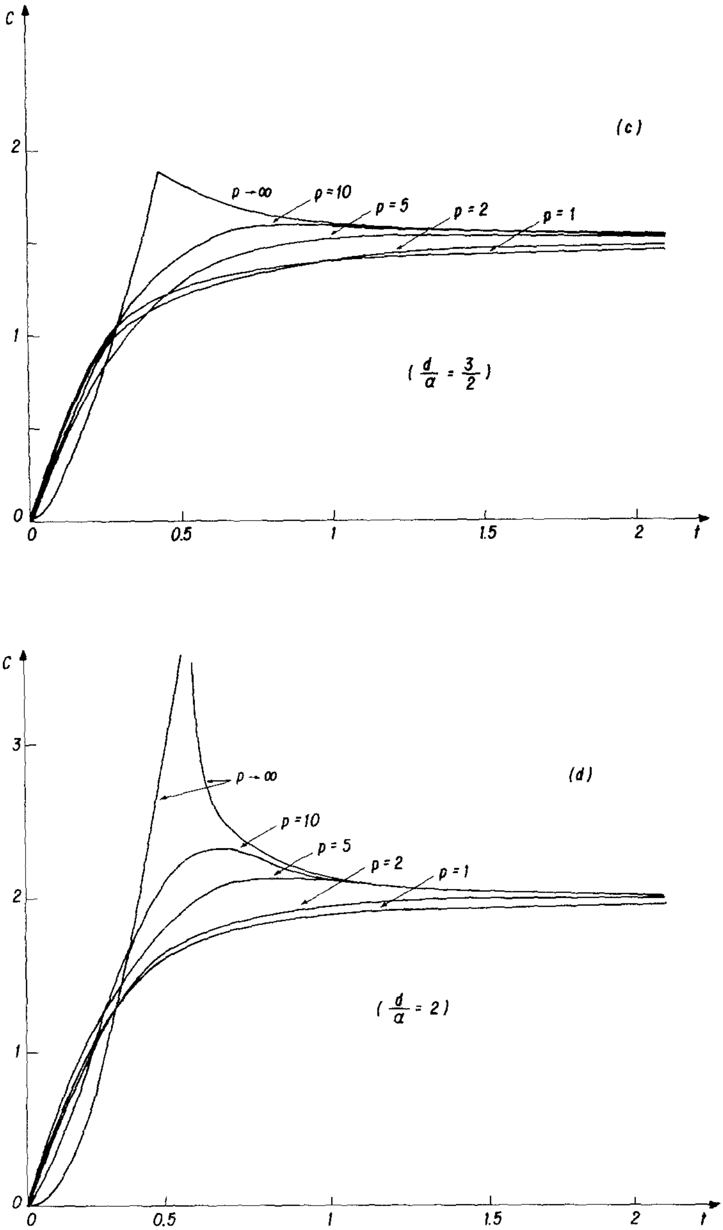
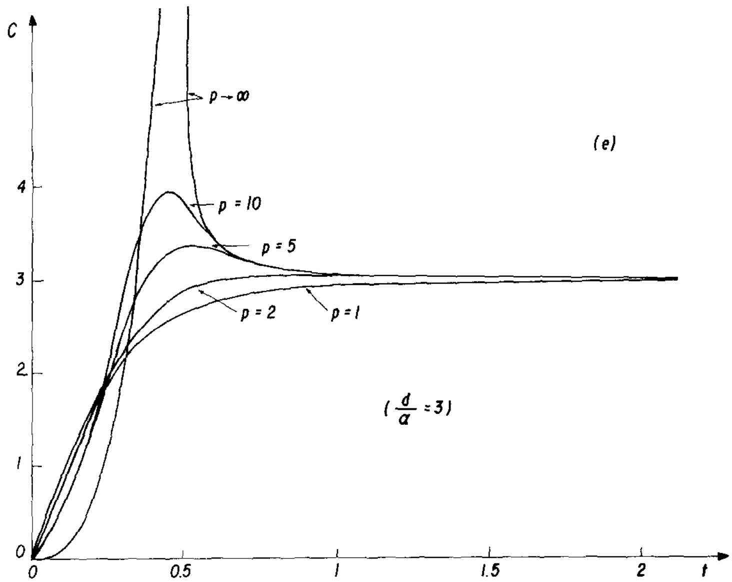


Fig. 4 (*continued*)



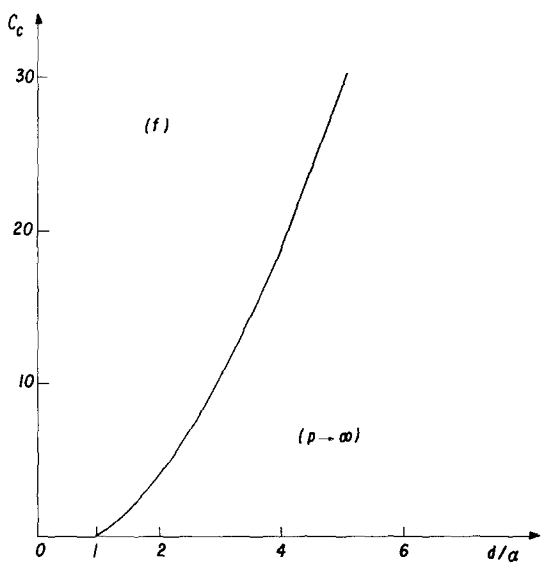


Fig. 4 (*continued*)

This is an interesting result, since it shows that asymptotically $C \propto T$ *for all finite values of p and d/α*, thus generalizing the well-known result for quasi-free electrons in a metal ($d=3$, $\alpha=2$, and $p=1$).

In the $p \to \infty$ limit, Eq. (31) becomes

$$u(t) = \frac{d}{\alpha} t^{d/\alpha + 1} \int_0^\infty dy \frac{y^{d/\alpha}}{e^y - 1}, \qquad t < t_c \tag{35}$$

where we have used that $\bar{\mu} \equiv 0$ if $t < t_c$. By using a result of Ref. 11 (p. 325), we can rewrite this equation as follows:

$$u(t) = \frac{d}{\alpha} \Gamma\left(\frac{d}{\alpha} + 1\right) \zeta\left(\frac{d}{\alpha} + 1\right) t^{d/\alpha + 1} \tag{36}$$

Hence

$$c = \frac{d}{\alpha} \Gamma\left(\frac{d}{\alpha} + 2\right) \zeta\left(\frac{d}{\alpha} + 1\right) t^{d/\alpha} \tag{37}$$

For the case $d/\alpha > 1$, this expression can be rewritten as follows:

$$c = \frac{d}{\alpha}\left(\frac{d}{\alpha} + 1\right) \frac{\zeta(d/\alpha + 1)}{\zeta(d/\alpha)} \left(\frac{t}{t_c}\right)^{d/\alpha} \tag{37'}$$

where we have used Eq. (22′). The $C \propto T^{d/\alpha}$ law we have obtained generalizes the Debye law for acoustic phonons ($d=3$, $\alpha=1$) as well as the $T^{3/2}$ law for magnons in the standard Heisenberg ferromagnet ($d=3$, $\alpha=2$).

Let us now focus on the high-temperature behavior of c. The integral of Eq. (31) can be treated in the same way as for Eq. (17); we obtain

$$c \sim \frac{d}{\alpha} \left\{ 1 + \frac{d/\alpha - 1}{2^{d/\alpha + 1} \Gamma(d/\alpha + 1) t^{d/\alpha}} - \frac{p(pd/\alpha - 1)}{(p+1)^{d/\alpha} [\Gamma(d/\alpha + 1)]^p \, t^{pd/\alpha}} \right\}, \qquad t \to \infty \tag{38}$$

Note that in the Fermi–Dirac (Bose–Einstein) statistics, the specific heat approaches the classical d/α value from below (above) when $d/\alpha > 1$, and the opposite happens when $d/\alpha < 1$; for $d/\alpha = 1$, the approach occurs from below for all values of p. Consequently, C versus T presents, for $d/\alpha > 1$, a maximum for p *high* enough (the maximum becomes a cusp in the $p \to \infty$ limit); for $d/\alpha < 1$, it presents a maximum for p *low* enough.

Let us now deduce the equation of state. The grand canonical partition function $\Xi_{\mathbf{k}}$ associated with the wave vector **k** is given by

$$\begin{aligned}\Xi_{\mathbf{k}} &= 1 + \exp[-\beta(\varepsilon_{\mathbf{k}} - \mu)] + \cdots + \exp[-p\beta(\varepsilon_{\mathbf{k}} - \mu)] \\ &= \frac{1 - \exp[-(p+1)\,\beta(\varepsilon_{\mathbf{k}} - \mu)]}{1 - \exp[-\beta(\varepsilon_{\mathbf{k}} - \mu)]}\end{aligned} \tag{39}$$

The total partition function Ξ equals $\prod_{\mathbf{k}} \Xi_{\mathbf{k}}$; consequently,

$$\ln \Xi = \int_0^\infty d\varepsilon\, \rho(\varepsilon) \ln \frac{1 - e^{-(p+1)\beta(\varepsilon - \mu)}}{1 - e^{-\beta(\varepsilon - \mu)}} \tag{40}$$

We know from thermodynamics that the pressure P satisfies

$$\beta P V = \ln \Xi$$

Hence

$$\begin{aligned}\beta P V &= \int_0^\infty d\varepsilon\, \rho(\varepsilon) \ln \frac{1 - e^{-(p+1)\beta(\varepsilon - \mu)}}{1 - e^{-\beta(\varepsilon - \mu)}} \\ &= \frac{\beta^{-d/\alpha}}{\varepsilon_0^{d/\alpha}} \int_0^\infty dy\, y^{d/\alpha - 1} \ln \frac{1 - e^{-(p+1)(y - \beta\mu)}}{1 - e^{-(y - \beta\mu)}}\end{aligned} \tag{41}$$

where we have used Eq. (5). On the other hand, also from thermodynamics, we have that

$$\begin{aligned}U &= -\left.\frac{\partial \ln \Xi}{\partial \beta}\right|_{\text{fixed } \beta u} \\ &= -\left.\frac{\partial(\beta P V)}{\partial \beta}\right|_{\text{fixed } \beta\mu} \\ &= \frac{d}{\alpha} \frac{\beta^{-d/\alpha - 1}}{\varepsilon_0^{d/\alpha}} \int_0^\infty dy\, y^{d/\alpha - 1} \ln \frac{1 - e^{-(p+1)(y - \beta\mu)}}{1 - e^{-(y - \beta\mu)}}\end{aligned}$$

And by using again Eq. (41), we obtain

$$P = (\alpha/d)\, U/V \tag{42}$$

which is exact for *all values of* p. Equation (42) transforms the analysis of the pressure into that of U, which we have already done.

Let us finally calculate the latent heat L (per particle) associated with the Bose–Einstein condensation (first-order) phase transition. The Clapeyron equation states that

$$L = T(dP/dT)_{T_c}(v_n - v_c) \tag{43}$$

where v_n (v_c) is the volume per particle in the *normal* (*condensed*) phase. But in the $N \gg 1$ limit, v_c vanishes[12] and v_n equals V/N; therefore

$$L = T\left(\frac{dP}{dT}\right)_{T_c} \frac{V}{N} = T\frac{\alpha}{d}\left[\frac{d(U/N)}{dT}\right]_{T_c} = k_B T \frac{\alpha}{d}\left(\frac{du}{dt}\right)_{t_c}$$

where we have used Eq. (42) and the definitions of u and t. By replacing Eq. (36) in this expression we obtain

$$L = k_B T \Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right)\left(\frac{d}{\alpha}+1\right) t_c^{d/\alpha}$$

and, by using Eq. (22′), we finally obtain

$$\frac{L}{k_B T} = \frac{\zeta(d/\alpha+1)}{\zeta(d/\alpha)}\left(\frac{d}{\alpha}+1\right) \tag{44}$$

$$\sim \begin{cases} \frac{\pi^2}{3}(d/\alpha - 1) & \text{if } \ d/\alpha - 1 \to +0 \\ d/\alpha + 1 & \text{if } \ d/\alpha \to \infty \end{cases} \tag{44′}$$

The dependence of L on d/α is represented in Fig. 5.

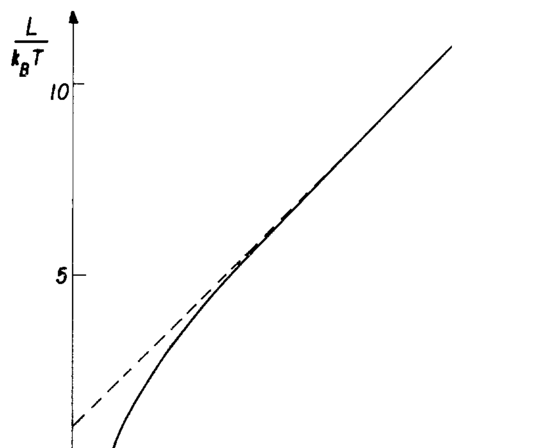


Fig. 5. The d/α dependence of the latent heat L per particle (in units of $k_B T$) associated with the Bose–Einstein condensation (first-order phase transition).

4. UNFIXED TOTAL NUMBER OF PARTICLES (QUASIPARTICLES)

In the present situation N is unfixed, and therefore μ vanishes for all temperatures and for all values of p. The average number $\langle N \rangle$ of particles is given by

$$\langle N \rangle = \int_0^\infty d\varepsilon\, \rho(\varepsilon) \left[\frac{1}{e^{\beta\varepsilon} - 1} - \frac{p+1}{e^{(p+1)\beta\varepsilon} - 1} \right] \tag{45}$$

Hence

$$\frac{\langle N \rangle}{V} = \frac{1}{V} \left(\frac{k_B T}{\varepsilon_0} \right)^{d/\alpha} \int_0^\infty dy\, y^{d/\alpha - 1} \left[\frac{1}{e^y - 1} - \frac{p+1}{e^{(p+1)y} - 1} \right]$$

where we have used Eq. (5). It follows that for $d/\alpha > 1$

$$\frac{\langle N \rangle}{V} = \frac{\zeta(d/\alpha)\, \Gamma(d/\alpha + 1)}{2^d \pi^{d/2} \Gamma(d/2 + 1) (\prod_{j=1}^d a_j)^{1/2}} \left[1 - \frac{1}{(p+1)^{d/\alpha - 1}} \right] (k_B T)^{d/\alpha} \tag{46}$$

where we have used Eq. (4').

Let us now focus on the internal energy. It is given by

$$U = \int_0^\infty d\varepsilon\, \varepsilon \rho(\varepsilon) \left[\frac{1}{e^{\beta\varepsilon} - 1} - \frac{p+1}{e^{(p+1)\beta\varepsilon} - 1} \right] \tag{47}$$

Hence

$$\frac{U}{V} = \frac{1}{V} \frac{(k_B T)^{d/\alpha + 1}}{\varepsilon_0^{d/\alpha}} \int_0^\infty dy\, y^{d/\alpha} \left[\frac{1}{e^y - 1} - \frac{p+1}{e^{(p+1)y} - 1} \right]$$

Therefore, for $d/\alpha > 1$, we obtain

$$\frac{U}{V} = \frac{\zeta(d/\alpha + 1)\, \Gamma(d/\alpha + 1) d/\alpha}{2^d \pi^{d/2} \Gamma(d/2 + 1) (\prod_{j=1}^d a_j)^{1/2}} \left[1 - \frac{1}{(p+1)^{d/\alpha - 1}} \right] (k_B T)^{d/\alpha + 1} \tag{48}$$

where we have used again Eq. (4'). The present $U/V \propto T^{d/\alpha + 1}$ law generalizes the well-known blackbody T^4 Stefan–Boltzmann law ($d = 3$, $\alpha = 1$, $p \to \infty$). The generalized radiation pressure can be obtained by replacing Eq. (48) into Eq. (42), which still holds.

5. CONCLUSION

In a unified framework, we have treated the parastatistics of an ideal confined (fixed-volume) d-dimensional gas of N particles (or quasiparticles) whose energy spectrum is given by Eq. (1) [which contains the important isotropic case $\varepsilon \propto |\mathbf{k}|^\alpha$ $(\alpha > 0)$]; each state can be occupied at most by p particles. We have studied, for arbitrary d, α, p, and T, and for both fixed and unfixed N cases, various thermodynamic quantities (chemical potential, average populations, internal energy, specific heat, pressure, and latent heat). In what follows we summarize the main results.

(i) The density of states satisfies $\rho(\varepsilon) \propto \varepsilon^{d/\alpha - 1}$. This form will imply that the thrmal dependence of all equilibrium thermodynamic quantities, and for all p, will depend on d and α on through the ratio d/α. In fact, most of the results summarized below remain unchanged (except for the prefactors) if $\rho \propto \varepsilon^{d/\alpha - 1}$ (i.e., if $\varepsilon \propto |\mathbf{k}|^\alpha$) in the $\varepsilon \to 0$ limit only, and not necessarily for all values of ε.

(ii) The Bose–Einstein condensation appears, for $d/\alpha > 1$, as a nonuniform convergence in the $p \to \infty$ limit, in complete analogy with phase transitions in interacting systems, which appear as nonuniform convergences in the thermodynamic limit $N \to \infty$. We obtain the explicit dependence of the critical temperature T_c on d/α [Eq. (22′)]: it is $T_c \propto (d/\alpha - 1)$ for $d/\alpha \gtrsim 1$, and $T_c \propto \alpha/d$ for $d/\alpha \gg 1$. The macroscopic population of the ground state satisfies $N_0/N = 1 - (T/T_c)^{d/\alpha}$ for $T \leqslant T_c$ and $d/\alpha > 1$. The latent heat L is obtained as an explicit function of d/α [Eq. (44)]; it satisfies $L/T \propto d/\alpha - 1$ for $d/\alpha \gtrsim 1$, and $L/T \sim d/\alpha + 1$ for $d/\alpha \gg 1$. The picture that emerges [here and in paragraphs (iv) and (v) which follow] is that the $p \to \infty$ limit is deeply different from any other case (i.e., $0 < p < \infty$). Moreover, all *finite* p paragases are qualitatively similar, and are well characterized by the Fermi–Dirac case ($p = 1$). Within this context, the view developed in Ref. 2, where the $d = 3$, $\alpha = 2$ Bose–Einstein condensation is referred for *all* values of p, can be considered as deeply misleading.

(iii) For fixed N, finite p, and low temperature, the chemical potential presents a quadratic departure from the $T = 0$ value [Eq. (26)]; the curvature changes its sign at $d/\alpha = 1$. In the high-temperature regime the first two (or three for $p > 1$) dominant terms (of the chemical potential) do not depend on p.

(iv) For fixed N, finite p, arbitrary d/α, and low temperature, the specific heat satisfies $C \propto T$, thus generalizing the standard result for quasi-free electrons in a conductor ($d = 3$, $\alpha = 2$, $p = 1$). In the $p \to \infty$ limit we obtain, for both fixed and unfixed N cases, $C \propto T^{d/\alpha}$ thus generalizing the Debye law for acoustic phonons in a crystal ($d = 3$, $\alpha = 1$) and the $T^{3/2}$ law for magnons in a Heisenberg ferromagnet ($d = 3$, $\alpha = 2$).

(v) For fixed N, arbitrary p and d/α, and high temperature, C approaches the classical value $(d/\alpha) N k_B$. For p low (high) enough, C

approaches this values from below (above) when $d/\alpha > 1$, and the opposite happens when $d/\alpha < 1$; consequently, C presents, for $d/\alpha > 1$, a maximum for p *high* enough (*cusp* in the $p \to \infty$ limit) and presents, for $d/\alpha < 1$, a maximum for p *low* enough.

(vi) For arbitrary p, d/α, and T and for both fixed and unfixed N cases, the pressure P and the internal energy U are related through $U = (d/\alpha) PV$ ($V \equiv$ volume), thus generalizing the standard result $U = (3/2) PV$ for $d = 3$, $\alpha = 2$, and $p = 1, \infty$.

(vii) For unfixed N, arbitrary p, d/α and T, the density of the internal energy (proportional to the power irradiated per unit area of the confining box) satisfies $U/V \propto T^{d/\alpha + 1}$ [Eq. (48)], thus generalizing the T^4 Stefan–Boltzmann law ($d = 3$, $\alpha = 1$, $p \to \infty$).

Also, since the 1968 paper by Gunton and Buckingham[9] (see also Ref. 13 and references therein), the Bose–Einstein condensation has been known to be related to the criticality of the spherical model (n-vector model with $n \to \infty$). Consistently, the fact that T_c should vanish in the $d/\alpha \to 1$ limit can be inferred from this relationship.

Finally, as a closing remark, let us recall that several studies[14–17] are available in which the bosonic limit ($p \to \infty$) is analyzed under a variety of conditions (for both spinless and magnetized bosons). By following along the lines of the present paper, it would be interesting to extend those results to arbitrary values of p.

APPENDIX. GENERALIZED SOMMERFELD INTEGRAL

Here we generalize to an arbitrary finite p the standard Sommerfeld integral for the FD statistics and its low-temperature expansion. We consider the integral

$$I(z) \equiv \int_{-\infty}^{\infty} dy\, H(y)\, f(y; z) \tag{A.1}$$

where

$$f(y; z) \equiv \frac{1}{e^{y-z} - 1} - \frac{p+1}{e^{(p+1)(y-z)} - 1}, \qquad p > 0 \tag{A.2}$$

To characterize the function $H(y)$, we introduce

$$K(y) \equiv \int_{-\infty}^{y} dy'\, H(y') \tag{A.3}$$

$K(y)$ is assumed to satisfy the following requirements:

(i) $$\lim_{y \to -\infty} K(y) = 0 \tag{A.4}$$

Hence

$$H(y) = dK(y)/dy \tag{A.5}$$

(ii) $$\lim_{y \to \infty} f(y; z)\, K(y) = 0 \tag{A.6}$$

(iii) $K(y)$ is analytic at $y = z$ (A.7)

The conditions are almost always satisfied in the physical situations, since typical functions $H(y)$ identically vanish for $y < 0$, asymptotically behave as a power law in the $y \to \infty$ limit, and are soft functions at $y = z$ [$H(y)$ might present singularities, but they are normally integrable and located at places such as $y = 0$, and not at $y = z$].

Integrating (A.1) by parts, we obtain

$$\begin{aligned} I(z) &= K(y)\, f(y; z)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dy\ K(y) \frac{df(y; z)}{dy} \\ &= -\int_{-\infty}^{\infty} dy\, K(y) \frac{df(y; z)}{dy} \end{aligned} \tag{A.8}$$

where we have used conditions (A.4) and (A.6) and the fact that $\lim_{y \to -\infty} f(y; z) = p$. By expanding $K(y)$ at $y = z$, we obtain

$$\begin{aligned} I(z) &= -K(z) \int_{-\infty}^{\infty} dy \frac{df}{dy} - \sum_{n=1}^{\infty} \left\{ \frac{1}{n!} \left[\frac{d^n K(y)}{dy^n} \right]_{y=z} \int_{-\infty}^{\infty} dy\, (y - z)^n \frac{df}{dy} \right\} \\ &= pK(z) - \sum_{m=1}^{\infty} \left\{ \frac{1}{(2m)!} \left[\frac{d^{2m} K(y)}{dy^{2m}} \right]_{y=z} \int_{-\infty}^{\infty} dy\, (y - z)^{2m} \frac{df}{dy} \right\} \end{aligned} \tag{A.9}$$

where we have used the fact that

$$\int_{-\infty}^{\infty} dy \frac{df}{dy} = -p$$

and that df/dy is an *even* function of $(y - z)$, and therefore all *odd* terms in the sum over n vanish. By using the connection between $K(y)$ and $H(y)$, we can rewrite Eq. (A.9) as follows:

$$I(z) = p \int_{-\infty}^{z} H(y)\, dy + \sum_{m=1}^{\infty} \left\{ \left[\frac{d^{2m-1} H(y)}{dy^{2m-1}} \right]_{y=z} a_m \right\} \tag{A.10}$$

with

$$a_m \equiv \frac{1}{(2m)!}\int_{-\infty}^{\infty} dx\, x^{2m} \frac{d}{dx}\left[\frac{p+1}{e^{(p+1)x}-1} - \frac{1}{e^x-1}\right]$$

$$= \frac{1}{(2m!)}\left[1 - \frac{1}{(p+1)^{2m-1}}\right]\int_{-\infty}^{\infty} dx\, x^{2m} \frac{d}{dx}\frac{1}{1-e^x}$$

$$= \frac{1}{(2m)!}\left[1 - \frac{1}{(p+1)^{2m-1}}\right] 2^{2m}\int_{-\infty}^{\infty} dx \frac{x^{2m}}{\sinh^2 x}$$

$$= \frac{1}{(2m)!}\left[1 - \frac{1}{(p+1)^{2m-1}}\right](2\pi)^{2m}|B_{2m}| \qquad \text{(A.11)}$$

where in the last step we have used Ref. 11, p. 352; the B_{2m} are the Bernoulli numbers. By using the fact that $|B_{2m}| = (2m)!\,\zeta(2m)/(2^{2m-1}\pi^{2m})$ we can finally express a_m as follows:

$$a_m = 2\zeta(2m)\left[1 - \frac{1}{(p+1)^{2m-1}}\right] \qquad \text{(A.12)}$$

and consequently

$$\int_{-\infty}^{\infty} dy\, H(y)\, f(y;z)$$

$$= p\int_{-\infty}^{z} H(y)\,dy + 2\sum_{m=1}^{\infty}\left\{\zeta(2m)\left[1 - \frac{1}{(p+1)^{2m-1}}\right]\left[\frac{d^{2m-1}H(y)}{dy^{2m-1}}\right]_{y=z}\right\}$$

$$= p\int_{-\infty}^{z} H(y)\,dy + \frac{\pi^2}{3}\left(1 - \frac{1}{p+1}\right)H'(z)$$

$$+\frac{\pi^4}{45}\left[1 - \frac{1}{(p+1)^3}\right]H'''(z) + \cdots \qquad \text{(A.13)}$$

which is the generalization we were looking for.

Quite frequently $H(y)$ is defined as follows:

$$H(y) = \begin{cases} 0 & \text{if} \quad y < 0 \\ Ay^{\gamma} & \text{if} \quad y \geqslant 0 \end{cases} \qquad \text{(A.14)}$$

In such a case, Eq. (A.13) can be rewritten as follows:

$$\int_0^{\infty} dy\, H(y)\, f(y;z)$$

$$= Az^{\gamma+1}\left\{\frac{p}{\gamma+1} + 2\sum_{m=1}^{\infty}\frac{\zeta(2m)}{z^{2m}}\left[1 - \frac{1}{(p+1)^{2m-1}}\right]\prod_{j=0}^{2(m-1)}(\gamma-j)\right\}$$

(A.15)

Finally, if we identify $y = \varepsilon/k_B T$ and $z = \mu/k_B T$, Eq. (A.13) takes the form

$$\int_{-\infty}^{\infty} d\varepsilon\, H(\varepsilon)\, f(\varepsilon)$$

$$= p \int_{-\infty}^{\mu} H(\varepsilon)\, d\varepsilon + 2 \sum_{m=1}^{\infty} \left\{ \zeta(2m) \left[1 - \frac{1}{(p+1)^{2m-1}} \right] [\mu^{2m} H^{(2m-1)}(\mu)] \left(\frac{k_B T}{\mu} \right)^{2m} \right\}$$

$$= p \int_{-\infty}^{\mu} H(\varepsilon)\, d\varepsilon + \frac{\pi^2}{3} \left(1 - \frac{1}{p+1} \right) [\mu^2 H'(\mu)] \left(\frac{k_B T}{\mu} \right)^2 + \frac{\pi^4}{45} \left[1 - \frac{1}{(p+1)^3} \right] [\mu^4 H'''(\mu)] \left(\frac{k_B T}{\mu} \right)^4 + o\left[\left(\frac{k_B T}{\mu} \right)^6 \right] \quad \text{(A.16)}$$

where in the estimation of the rest, $o[(k_B T/\mu)^6]$, we have assumed the quite frequent fact that $H^{(n)}(\mu)$ is of the order of $H(\mu)/\mu^n$.

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